



HALL-LITTLEWOOD POLYNOMIALS AS RANDOMISATIONS

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We express the Hall-Littlewood polynomials, up to a multiplicative factor, as expectations of Schur functions whose indexing partition is a coordinatewise randomisation of the original partition by a particular sequence of explicit correlated random variables. We also treat the case of the spin Hall-Littlewood polynomials by first writing them as randomisations of the Hall-Littlewood polynomials and generalise this framework to a class of symmetric functions that include the Jing and Milne polynomials.

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1. Introduction

1.1. Motivations

The Hall-Littlewood polynomials are deformations of the Schur polynomials that depend on a parameter t and that interpolate between Schur functions, monomial functions [21, III-2, (2.3) & (2.4) p. 208] and Schur Q -functions [21, III-8, p. 250]. Their structure constants $c_{\lambda, \mu}^{\nu}(t)$ (i.e. the products appearing in the decomposition $P_{\lambda}P_{\mu} = \sum_{\nu} c_{\lambda, \mu}^{\nu} P_{\nu}$) are the Hall polynomials that deform the Littlewood-Richardson coefficients and are equal to the number of subgroups of type λ and cotype μ in a finite abelian p -group of type ν with $p = t^{-1}$ [21, II & III]. There has been an extensive study of these polynomials and we refer to Macdonald [21] for some properties.

More recently, these polynomials were used in a probabilistic way, within the theory of *integrable probability* [3, 4] that is a deformation of the more classical theory of *determinantal probability* (the study of determinantal point processes). The probabilistic objects here defined are probability measures or Markov chains on the set of partitions of an integer. For instance, the Schur measure [25] is defined by

$$\mathbb{P}_{\text{Schur}(X, Y)}(\{\lambda\}) := H[-XY] s_{\lambda}[X] s_{\lambda}[Y] \quad (1)$$

where s_{λ} is a Schur function and H is defined in (9). Here, X and Y are two *positive* specialisations of the Schur function (i.e. algebra morphisms of the space $\mathbf{\Lambda} = \mathbb{C}[p_k]_{k \geq 1}$ of symmetric functions to $\mathbb{R}_+$). The normalisation constant $H[-XY]$ is here to ensure that one has a probability measure, i.e., that $\sum_{\lambda} \mathbb{P}_{X, Y}(\{\lambda\}) = 1$.

A dynamical version $(\mathbb{P}_{\text{Schur}(X_1 + \dots + X_k, Y_1 + \dots + Y_k)})_{k \geq 1}$ with sequences of specialisations $(X_k, Y_k)_{k \geq 1}$ defining a particular Markov chain $(\lambda^{(k)})_{k \geq 1}$ on the set of partitions was also constructed in [26] (see § 2 for the sum of specialisations). Amongst the very interesting properties of such a measure/Markov chain lies the fact that its so-called correlation functions of the shifted parts $\rho_k(m_1, \dots, m_k) := \mathbb{P}_{\text{Schur}(X, Y)}(\lambda + \delta \supset \{m_1, \dots, m_k\})$ can be written with a determinant [25, eq. (1.1)].

Hall-Littlewood polynomials were similarly used to define the Hall-Littlewood measure in [2] (and more generally the Macdonald measure)

$$\mathbb{P}_{\text{HL}(X, Y|t)}(\{\lambda\}) := H[-(1-t)XY] P_{\lambda}[X|t] Q_{\lambda}[Y|t] \quad (2)$$

and were subsequently studied in several articles (see e.g. the surveys [3, 4] for a non-exhaustive study, and [8] for a recent such paper).

So far, no probabilistic link was given between the correlation functions of the Schur measure and the ones of the Hall-Littlewood measure, and no probabilistic link was also given at the level of the functions themselves. The goal of this short note is to provide a link between the functions (leaving the correlation functions for a future study). We will also study a natural modification of the Hall-Littlewood polynomials that was introduced by Borodin and Petrov [5] and were subsequently called the *Borodin-Petrov polynomials* (or more recently *spin-Hall-Littlewood polynomials*), see § 2.2.5 for a precise definition. Probability theory will appear as very relevant in understanding the nature of the deformation allowing to pass from one function to another.

1.2. Main results

1.2.1 Hall-Littlewood Polynomials

We are using the notations of § 2. The main theorem that we prove is the following:

Theorem 1 (Hall-Littlewood functions as randomisations). *For all $t \in (0, 1)$, we have:*

- *For the Q -Hall-Littlewood polynomials*

$$Q_\lambda(X|t) = (1-t)^{-\binom{\ell(\lambda)}{2}} \mathbb{E}(\tilde{s}_{\lambda-\mathcal{G}(t)}[(1-t)X]) \quad (3)$$

where $\tilde{s}_\lambda(X)$ is defined in (17) for all $\lambda \in \mathbb{Z}^\ell$ and is equal to $s_\lambda(X)$ when λ is a (skew) partition. The random vector $\mathcal{G}(t)$ is given in (29).

- *For the R -Hall-Littlewood polynomials*

$$R_\lambda(x_1, \dots, x_N|t) = (1+t)^{\binom{N}{2}} \mathbb{E}\left((-1)^{\mathcal{B}(t)} \hat{s}_{\lambda-\mathcal{B}(t)}(X)\right) \quad (4)$$

where $\mathcal{B}(t)$ and $\mathcal{B}(t)$ are defined in (30) and $\hat{s}_\lambda$ is defined in (18) for all $\lambda \in \mathbb{Z}^\ell$ and is equal to $s_\lambda(X)$ when λ is a (skew) partition.

Remark 2. Defining $\sigma_t^2 := -\ln(1-t) > 0$ and $G \sim \mathcal{N}(0, 1)$ independent of $\mathcal{G}(t)$, one has $e^{\sigma^2 \ell^2 / 2} = \mathbb{E}(e^{\ell G})$ hence

$$Q_\lambda(X|t) = \mathbb{E}\left(e^{(\sigma_t G + \sigma_t^2 / 2)\ell(\lambda)} \tilde{s}_{\lambda-\mathcal{G}(t)}[(1-t)X]\right). \quad (5)$$

Remark 3. With the notations of § 2.2.4, the Hall-Littlewood measure (2) becomes

$$\mathbb{P}_{\text{HL}(X,Y|t)}(\{\lambda\}) = \frac{(1+t)^{\binom{N}{2}}}{v_\lambda(t)}$$

$$\times \mathbb{E}((-1)^{\mathbf{B}(t)} e^{(\sigma_t G + \frac{\sigma_t^2}{2})\ell(\lambda)} \times \frac{\tilde{s}_{\lambda-\mathbf{g}(t)}[(1-t)X] \hat{s}_{\lambda-\mathbf{B}(t)}[Y]}{H[(1-t)XY]}).$$

Inside the expectation is a more general measure $s_{\lambda-\mu}[\mathcal{A}]s_{\lambda-\nu}[\mathcal{B}]/H[\mathcal{AB}]$ with a skew Schur function (once the vector ordered). The term $v_\lambda(t)^{-1}$ nevertheless prevents from computing directly its correlation functions using the Schur measure ones and an additional manipulation is required for that.

1.2.2 The Spin Hall-Littlewood/Borodin-Petrov Polynomials

These particular symmetric functions introduced by Borodin and Petrov [5] and studied by Borodin [1] enjoy the same type of probabilistic writing as the Hall-Littlewood polynomials.

Theorem 4 (spin Hall-Littlewood functions as randomisations). *We have*

$$\begin{aligned} \tilde{F}_\lambda(X|t, (p, \kappa)) &= \mathbb{E}(R_{\mathbf{Z}_\lambda(p, \kappa)}[X|t]) \\ &= (1+t)^{\binom{N}{2}} \mathbb{E}((-1)^{\mathbf{B}(t)} \hat{s}_{\mathbf{Z}_\lambda(p, \kappa)-\mathbf{B}(t)}[X]) \end{aligned} \quad (6)$$

where $\tilde{F}_\lambda$ is the probabilistic spin Hall-Littlewood/Borodin-Petrov polynomial defined in (23) and the random vector $\mathbf{Z}_\lambda(p, \kappa)$ is defined in (31).

A corollary of this last result is the following rediscovery of a result of Bufetov-Mucciconi-Petrov [7, Thm. 8.2, (8.5)]:

Corollary 5 (Diagonal action of the Macdonald operators on spin Hall-Littlewood functions). *We have for $X = \{x_1, \dots, x_n\}$*

$$\mathcal{D}(u|0, t) \tilde{F}_\lambda(X|t, (p, \kappa)) = H\left[-\varepsilon u \frac{1-t^n-\ell(\lambda)}{1-t}\right] \tilde{F}_\lambda(X|t, (p, \kappa)) \quad (7)$$

where $\mathcal{D}(u|0, t)$ is the Macdonald operator defined in (20).

1.3. Organisation of the paper

The plan of the article is as follows: in § 2 we gather several notations and concepts that are used throughout the article (generalities in § 2.1, symmetric functions in § 2.2 and probability theory in § 2.3). In § 3, we prove theorem 1 (§ 3.1), theorem 4 and corollary 5 (§ 3.2) ; we conclude in § 3.3 with some natural generalisations of the study to other types of symmetric functions such as the Jing or the Milne symmetric functions.

2. Conventions and Notations

2.1. General notations

We will use the notation

$$[z^n]f(z) = a_n \quad \text{if} \quad f(z) = \sum_n a_n z^n$$

to denote the n -th (Fourier) coefficient of a Laurent series f . Such a coefficient can be understood a Fourier integral $\oint_{\rho\mathbb{U}} z^{-n} f(z) \frac{dz}{2i\pi z}$ for ρ stricly less than the radius of convergence of f . We also use the multi-index notation

$$X^\alpha := \prod_{k \geq 1} x_k^{\alpha_k} \tag{8}$$

and the notation $[X^\alpha]f(X)$ for the multivariate Fourier coefficient.

2.2. Symmetric functions

2.2.1 Generalities

The partitions of an integer n are weakly decreasing sequences $\lambda := (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m)$ such that $|\lambda| := \sum_{i=1}^m \lambda_i = n$. The length m of λ will be denoted by $\ell(\lambda)$. We will also denote by $m_k(\lambda) := \sum_{j \geq 1} \mathbb{1}_{\{\lambda_j = k\}}$ the number of parts of λ equal to k . Note that $\sum_{k=1}^{\lambda_1} m_k(\lambda) = \ell(\lambda)$ and $\sum_{k \geq 1} k m_k(\lambda) = |\lambda|$. We refer to [21, ch. I.1] for generalities concerning them.

For notations and definitions concerning symmetric functions, the reference is Macdonald [21]. We denote by p_k the power functions, h_k the homogeneous complete symmetric functions and e_k the elementary symmetric functions. The ring of symmetric functions is denoted by $\mathbf{\Lambda} := \mathbb{C}[p_k]_{k \geq 1} = \mathbb{C}[h_k]_{k \geq 0} = \mathbb{C}[e_k]_{k \geq 0}$.

The use of λ -ring/plethystic notations and alphabets will be done in the same vein as [12, 19, 20], using in particular the convention of [12, § 2] that writes plethysm with a bracket $[\cdot]$: the plethysm of a symmetric function f in an abstract alphabet $\mathcal{A}$ will be denoted $f[\mathcal{A}]$. In the λ -ring vision of e.g. [19, 20], symmetric functions are polynomial functors in the vein of [21, Appendix A p. 149]. As a result, the term functor can be used to designate such quantities.

2.2.2 Plethystic operations

Given the sets of variables or “alphabets” $X := \{x_k\}_{k \geq 1}$ and $Y := \{y_k\}_{k \geq 1}$, the power functions are defined by $p_k(x_1, x_2, \dots) := \sum_{\ell \geq 1} x_\ell^k$. We define the sum $X + Y$ and the product XY by

$$\begin{aligned} p_k[X + Y] &= p_k[X] + p_k[Y] \\ p_k[XY] &= p_k[X]p_k[Y]. \end{aligned}$$

Equivalently,

$$X + Y = \{x_k, y_k\}_{k \geq 1}, \quad XY = \{x_i y_j\}_{i, j \geq 1}.$$

Note the difference of convention between $p_k(X + Y) = \sum_{\ell} (x_\ell + y_\ell)^k$ and $p_k[X + Y]$.

We define the generating series of the complete homogeneous symmetric functions by

$$H[X] := \sum_{n \geq 0} h_n(X) = \prod_{k \geq 1} \frac{1}{1 - x_k}. \quad (9)$$

It clear that

$$H[X + Y] = H[X]H[Y]. \quad (10)$$

The difference of two alphabets $X - Y$ is defined as the formal alphabet such that [21, 1-3, ex. 23]

$$H[X - Y] = \frac{H[X]}{H[Y]}.$$

This last definition is equivalent to [21, I (2.10)]

$$p_k[X - Y] = p_k[X] - p_k[Y].$$

Note that $f(-X) := f(-x_1, -x_2, \dots) \neq f[-X]$. To differentiate between these two operations, we define the “minus sign” alphabet

$$\varepsilon := \{-1\} \quad (11)$$

which is such that $f[\varepsilon X] = f(-X)$.

We define the following *root* alphabets

$$X^{\varepsilon R} := \{x_j x_i^{-1}, 1 \leq i < j \leq n\}, \quad X^R := \{x_i x_j^{-1}, 1 \leq i < j \leq n\}. \quad (12)$$

Setting the canonical basis of $\mathbb{R}^n$ to be $(e_i)_i$, R , noted R_0^+ in [21, I-5 ex. 16], is then the set of *positive roots of type A_{n-1}* :

$$R := \{e_i - e_j, j > i\}.$$

The set of *negative roots of type A_{n-1}* is defined as $\varepsilon R = \{\varepsilon r, r \in R\}$ using the convention (11).

With the multi-index notation, $X^R = \{X^r, r \in R\} = \{x_j/x_i, i > j\}$ which explains (12).

One has for instance

$$H[X^R] = \prod_{1 \leq i < j} \frac{1}{1 - x_i x_j^{-1}}, \quad H[-X^{\varepsilon R}] = \prod_{1 \leq i < j} (1 - x_j x_i^{-1}).$$

The alphabets $1 - t$ and $\frac{1}{1-q} = \sum_{k \geq 0} q^k$ are defined by

$$p_k[1 - t] = 1 - t^k, \quad p_k\left[\frac{1}{1-q}\right] = \frac{1}{1 - q^k}. \quad (13)$$

2.2.3 Schur Functions

The Schur functions are denoted by s_λ and have several equivalent definitions. One could define it with the following ratio of *alternants* [21, ch. I-3, eq. (3.1)]

$$s_\lambda(x_1, \dots, x_n) = \frac{\det\left(x_i^{\lambda_j + n - j}\right)_{1 \leq i, j \leq n}}{\Delta(x_1, \dots, x_n)}, \quad \Delta(X) = \prod_{i < j} (x_i - x_j). \quad (14)$$

Another definition is the Jacobi-Trudi identity [21, ch. I-3, eq. (3.4)]

$$s_\lambda = \det(h_{\lambda_i + j - i})_{1 \leq i, j \leq \ell}, \quad \text{for all } \ell \geq \ell(\lambda) \quad (15)$$

which is equivalent, with simple manipulations, to the Fourier coefficient form

$$s_\lambda[X] = [U^\lambda] H[XU - U^{\varepsilon R}], \quad U := \{u_1, \dots, u_\ell\}, \quad \text{for all } \ell \geq \ell(\lambda). \quad (16)$$

Indeed, using the Vandermonde determinant $\Delta(U)$ and the vector

$$\delta_\ell := (\ell - 1, \ell - 2, \dots, 1, 0) \in \mathbb{N}^\ell$$

one has $H[-U^{\varepsilon R}] = U^{-\delta_\ell} \Delta(U)$ and

$$\begin{aligned} [U^\lambda] H[XU - U^{\varepsilon R}] &= [U^\lambda] H[XU] H[-U^{\varepsilon R}] = [U^\lambda] \prod_{i=1}^{\ell} H[u_i X] U^{-\delta_\ell} \Delta(U) \\ &= [U^{\lambda+\delta_\ell}] \prod_{i=1}^{\ell} H[u_i X] \sum_{\sigma \in \mathfrak{S}_\ell} \varepsilon(\sigma) \prod_{r=1}^{\ell} u_r^{\ell-\sigma(r)} \\ &= \sum_{\sigma \in \mathfrak{S}_\ell} \varepsilon(\sigma) \prod_{r=1}^{\ell} [u^0] u^{-\lambda_r - \ell + r} u^{\ell-\sigma(r)} H[uX] \\ &= \sum_{\sigma \in \mathfrak{S}_\ell} \varepsilon(\sigma) \prod_{r=1}^{\ell} h_{\lambda_r + \sigma(r) - r}[X] \\ &= \det(h_{\lambda_i + j - i}[X])_{1 \leq i, j \leq \ell}. \end{aligned}$$

In view of (16), we define the following function indexed by $\lambda \in \mathbb{Z}^\ell$:

$$\tilde{s}_\lambda(X) := [Y^\lambda] H[XY - Y^{\varepsilon R}]. \quad (17)$$

Formula (17) is thus equal to $\pm$ (16) or to 0 if one coordinate of λ is negative. It is equal to a proper Schur function indexed by a (skew) partition and multiplied by a sign ± 1 using exchanges of lines/columns in the Toeplitz minor given by the Jacobi-Trudi formula (15). The process of transforming $\tilde{s}_\lambda$ into $\pm s_{\tilde{\lambda}}$ for a certain partition $\tilde{\lambda}$ is called *rectification* or *Littlewood modification rule* and amounts to “let gravity act” when the Young diagram of the partition is drawn with the French convention ; see e.g. the exposition by Rosas [28].

Another natural function to define in view of (14) is then, for all $\lambda \in \mathbb{Z}^n$ (i.e. $\ell(\lambda) \leq n$)

$$\hat{s}_\lambda(x_1, \dots, x_n) = \frac{\det(x_i^{\lambda_j + n - j})_{1 \leq i, j \leq n}}{\Delta(x_1, \dots, x_n)}. \quad (18)$$

Citing Macdonald [21, I-3, end p. 40]

Notice that the definition (14) makes sense for any integer vector $\lambda \in \mathbb{Z}^n$ such that $\lambda + \delta_n$ has no negative parts. If the numbers $\{\lambda_i + n - i\}_{1 \leq i \leq n}$ are not all distinct, then $s_\lambda = 0$. If they are all distinct, then we have $\lambda + \delta = w(\mu + \delta)$ for some $w \in S_n$, and some partition μ , and $s_\lambda = \varepsilon(w)s_\mu$.

Similarly to $\tilde{s}_\lambda$, one thus also has $\hat{s}_\lambda = a \times s_\mu$ with $a \in \{0, \pm 1\}$ and μ a partition.

2.2.4 Hall-Littlewood polynomials

The Hall-Littlewood polynomials are defined by

$$\begin{aligned}
 \bullet \quad Q_\lambda(X|t) &= [U^\lambda] H[(1-t)(XU - U^{\varepsilon R})] & [21, (2.15) \text{ p. 211}] \\
 \bullet \quad P_\lambda(X|t) &= \frac{1}{v_\lambda(t)} \sum_{\sigma \in \mathfrak{S}_n} \sigma \cdot (X^\lambda H[(1-t)X^{\varepsilon R}]) & [21, \text{p. 204 \& 207}] \\
 &=: \frac{1}{v_\lambda(t)} R_\lambda(X|t) & [21, (2.1) \text{ p. 208}] \\
 &= \frac{1}{b_\lambda(t)} Q_\lambda(X|t) & [21, (2.11) \text{ p. 210}]
 \end{aligned} \tag{19}$$

where [21, (2.12) p. 210]

$$\begin{aligned}
 b_\lambda(t) &:= \prod_{k \geq 1} \varphi_{m_k(\lambda)}(t), \quad \varphi_m(t) := (t; t)_m = \prod_{j=1}^m (1 - t^j) \\
 v_\lambda(t) &:= \frac{1}{(1-t)^{\ell(\lambda)}} b_\lambda(t) = \prod_{k \geq 1} \frac{\varphi_{m_k(\lambda)}(t)}{(1-t)^{m_k(\lambda)}} = \prod_{k \geq 1} \prod_{j=1}^{m_k(\lambda)} \frac{1-t^j}{1-t}.
 \end{aligned}$$

The Hall-Littlewood polynomial is a specialisation of the Macdonald polynomial $Q_\lambda(X|q, t)$ for $q = 0$ [21, VI-4, (4.14) (ii) p. 324]. Such polynomials are eigenfunctions of the following operator, with $X := \{x_1, \dots, x_n\}$ and $X^{-1} := \{x_1^{-1}, \dots, x_n^{-1}\}$ [21, VI-3, (3.5)_r p. 316]

$$\begin{aligned}
 \mathcal{D}^{(n)}(u|q, t) &:= \sum_{I \subset [n]} (-u)^{|I|} t^{\binom{|I|}{2}} a_t(X_I, X_{I^c}) \mathcal{T}_{(q-1)X_I}, \\
 a_t(X_I, X_{I^c}) &:= H\left[(1-t)X_I X_{I^c}^{-1}\right] \\
 &= \sum_{I \subset [n]} (-u)^{|I|} \frac{\Delta[X + (t-1)X_I]}{\Delta[X]} \mathcal{T}_{(q-1)X_I}
 \end{aligned}$$

$$= \sum_{I \subset [n]} (-u)^{|I|} \frac{\mathcal{T}_{(t-1)X_I} \Delta[X]}{\Delta[X]} \mathcal{T}_{(q-1)X_I}$$

where the *plethystic translation* is defined by

$$\mathcal{T}_A : f[X] \mapsto f[X + A]$$

and in particular

$$\mathcal{T}_{(q-1)X_I} : f[X] \mapsto f[X + (q-1)X_I].$$

Such an operator is an example of a q -difference operator and can be obtained from integrable (quantum mechanical) systems [9, 11, 23]. See also [13, 16–18, 24, 27, 30–34] for examples of difference operators of the same flavour coming from integrable systems and geometry.

In the case of the Hall-Littlewood polynomials,

$$\mathcal{T}_{-X_I} : f[X] \mapsto f[X - X_I] = f[X_I c]$$

and

$$\mathcal{D}^{(n)}(u|0, t) = \sum_{I \subset [n]} (-u)^{|I|} \frac{\mathcal{T}_{(t-1)X_I} \Delta[X]}{\Delta[X]} \mathcal{T}_{-X_I}. \quad (20)$$

The eigenvalue of $\mathcal{D}^{(n)}(u|0, t)$ acting on $Q_\lambda(X|t)$ is given by $H\left[-\varepsilon u \frac{1-t^{n-\ell(\lambda)}}{1-t}\right] = \prod_{j=0}^{n-\ell(\lambda)-1} (1 + ut^j) = (-u; t)_{n-\ell(\lambda)}$ [21, VI-4, (4.15) p. 324]

$$\mathcal{D}^{(n)}(u|0, t) Q_\lambda(X|t) = H\left[-\varepsilon u \frac{1-t^{n-\ell(\lambda)}}{1-t}\right] Q_\lambda(X|t). \quad (21)$$

2.2.5 Spin Hall-Littlewood/Borodin-Petrov Polynomials

The *Borodin-Petrov polynomials* or *spin Hall-Littlewood polynomials*¹ were originally defined in [5] and subsequently studied in [1]. They read for all λ such that $\ell(\lambda) \leq N$ and $X := \{x_1, \dots, x_N\}$ [1, (1.2)]

$$F_\lambda(x_1, \dots, x_N|t, s) := (1-t)^N H[sX] \times \sum_{\sigma \in \mathfrak{S}_N} \sigma \cdot \left(H[(1-t)X^{\varepsilon R}] \prod_{j=1}^{\ell(\lambda)} \left(\frac{x_j - s}{1 - sx_j} \right)^{\lambda_j} \right). \quad (22)$$

¹Depending on the conventions of normalisations, the polynomials can be replaced with rational fractions. We will perform a slight abuse of language and talk about a polynomial even when the factor required to turn the rational fraction into a polynomial is missing.

One immediately sees that for $s = 0$, one recovers $(1-t)^N R_\lambda(X|t)$ defined in (19). Note also that since $\ell(\lambda) \leq N$, one can replace the product $\prod_{j=1}^{\ell(\lambda)}$ by $\prod_{j=1}^N$ in (22). And since $H[sX]$ is symmetric, one can also write

$$F_\lambda(x_1, \dots, x_N | t, s) := (1-t)^N \sum_{\sigma \in \mathfrak{S}_N} \sigma \cdot \left(H[(1-t)X^{\varepsilon R}] \prod_{j=1}^N \frac{1}{1-sx_j} \left(\frac{x_j-s}{1-sx_j} \right)^{\lambda_j} \right).$$

The discussion by Borodin [1, introduction] shows that one can start from the general rational function $\frac{ax+b}{cx+d}$ and, using the diverse possible scalings, end up with $\frac{x-s}{1-sx}$. Let $0 < p < \kappa < 1$ and $\mathbf{s} := (p, \kappa)$. We will choose instead to define the *probabilistic* Borodin-Petrov/spin Hall-Littlewood polynomials by

$$\begin{aligned} \tilde{F}_\lambda(x_1, \dots, x_N | t, \mathbf{s}) &:= \left(\frac{1-\kappa}{1-p} \right)^{|\lambda|} \\ &\times \sum_{\sigma \in \mathfrak{S}_N} \sigma \cdot \left(H[(1-t)X^{\varepsilon R}] \prod_{j=1}^N \left(\frac{1-px_j}{1-\kappa x_j} \right)^{\lambda_j} \right). \end{aligned} \quad (23)$$

One can recover the functions given in (22) using

$$F_\lambda \left(\frac{1}{\sqrt{p\kappa}} X \middle| t, s = \sqrt{\frac{\kappa}{p}} \right) := (1-t)^N H[sX] \left(-s \frac{1-p}{1-\kappa} \right)^{|\lambda|} \tilde{F}_\lambda(X | t, (p, \kappa)).$$

Borodin [1, (7.8)] furnishes in addition a *dual* spin-Hall-Littlewood polynomial given by the following integral representation (up to a multiplicative factor) with $U := \{u_1, \dots, u_n\}$, $X := \{x_1, \dots, x_N\}$ and $N, n \geq \ell(\lambda)$:

$$\begin{aligned} G_\lambda(X | t, s) &= [U^0] H[(1-t)(UX - U^{\varepsilon R})] \\ &\times \prod_{j=1}^n \frac{1}{(1-su_j)(1-su_j^{-1})} \left(\frac{1-su_j}{u_j-s} \right)^{\lambda_j}. \end{aligned}$$

Here, the extraction of the coefficient is not understood anymore as a Fourier integral on a circle but instead as an iterated contour integral with a specific choice of contour: $[u_j](\cdot) := \oint_{\mathcal{C}} (\cdot) \frac{du_j}{2i\pi u_j}$ where $\mathcal{C}$ is such that (1) $q\mathcal{C} \subset \mathcal{C}$, (2) $\mathcal{C} \not\subset \{s^{-1}\} \cup X^{-1}$ and (3) $\left\{ s \frac{1-t^n}{1-t} \right\} := \{s, st, \dots, st^{n-1}\} \subset \mathcal{C}$ (see [6, fig. 6] for a picture).

One replaces this last function by its *probabilistic form*

$$\begin{aligned} \tilde{G}_\lambda(X|t, \mathbf{s}) &= \left(\frac{1-\kappa}{1-p} \right)^{|\lambda|} [U^0] H[(1-t)(UX - U^\varepsilon R)] \\ &\quad \times \prod_{j=1}^n \frac{(1-s)^2}{(1-su_j)(1-su_j^{-1})} \left(\frac{1-pu_j}{1-\kappa u_i} \right)^{\lambda_j}. \end{aligned} \quad (24)$$

One can take $n = \ell(\lambda)$ since the final result does not depend on $n \geq \ell(\lambda)$ in the same vein as for the Jacobi-Trudi formula (15) or the Hall-Littlewood Fourier coefficient form (16).

2.3. Probability theory

A geometric random variable is a random variable $G(t) \sim \text{Geom}(t)$ with $t \in (0, 1)$ whose law is given by

$$\mathbb{P}(G(t) = k) = t^k(1-t)\mathbb{1}_{\{k \geq 0\}}, \quad \text{for all } k \in \mathbb{Z}. \quad (25)$$

Its Fourier transform is given for all $z = e^{i\theta} \in \mathbb{U}$ by

$$\mathbb{E}\left(z^{G(t)}\right) = \frac{1-t}{1-tz}. \quad (26)$$

One also defines the random variable $Z(a, b)$ for $0 < a < b < 1$ by

$$\mathbb{E}\left(z^{Z(a,b)}\right) = \frac{1-b}{1-a} \frac{1-az}{1-bz}. \quad (27)$$

A Bernoulli random variable with parameter $p \in (0, 1)$ is $B(p) \sim \text{Ber}(p)$ and is defined by $B(p) \in \{0, 1\}$, and

$$\mathbb{P}(B(p) = 1) = p = 1 - \mathbb{P}(B(p) = 0). \quad (28)$$

It is not hard to see that $Z(a, b)$ has the same probability distribution as the random variable $G(b)B\left(\frac{1-a/b}{1-a}\right)$ with $G(b) \sim \text{Geom}(b)$ and where the random variables in the product are independent, what we denote by

$$\begin{aligned} Z(a, b) &\stackrel{\mathcal{L}}{=} G(b)B\left(\frac{1-a/b}{1-a}\right) \\ \iff \mathbb{P}(Z(a, b) = k) &= \mathbb{P}\left(G(b)B\left(\frac{1-a/b}{1-a}\right) = k\right), \text{ for all } k \in \mathbb{N}. \end{aligned}$$

3. Proof of Theorems 1 and 4

3.1. Hall-Littlewood case (theorem 1)

Proof:

- In the Q_λ case, one has for all $\ell \geq \ell(\lambda)$

$$\begin{aligned}
 Q_\lambda(x_1, \dots, x_N | t) &= [U^\lambda] H[(1-t)(XU - U^{\varepsilon R})] \\
 &= [U^\lambda] H[(1-t)XU - U^{\varepsilon R}] H[tU^{\varepsilon R}] \\
 &= [U^\lambda] H[(1-t)XU - U^{\varepsilon R}] \prod_{1 \leq i < j \leq \ell} \frac{1}{1 - tu_i u_j^{-1}} \\
 &= \frac{1}{(1-t)^{\binom{\ell}{2}}} [U^\lambda] H[(1-t)XU - U^{\varepsilon R}] \prod_{1 \leq i < j \leq \ell} \frac{1-t}{1 - tu_i u_j^{-1}}.
 \end{aligned}$$

Now, using (26), with an i.i.d. sequence $(G_{i,j}(t))_{1 \leq i, j \leq \ell}$ of $\text{Geom}(t)$ random variables, one has

$$\begin{aligned}
 \prod_{1 \leq i < j \leq \ell} \frac{1-t}{1 - tu_i u_j^{-1}} &= \prod_{1 \leq i < j \leq \ell} \mathbb{E}\left((u_i u_j^{-1})^{G_{i,j}(t)}\right) \\
 &= \mathbb{E}\left(\prod_{1 \leq i < j \leq \ell} u_i^{G_{i,j}(t)} u_j^{-G_{i,j}(t)}\right) \\
 &= \mathbb{E}\left(\prod_{i=1}^{\ell} u_i^{\mathcal{G}_i(t)}\right)
 \end{aligned}$$

where

$$\mathcal{G}_i(t) := \sum_{j=1}^{i-1} G_{i,j}(t) - \sum_{j=i+1}^{\ell} G_{j,i}(t), \quad \mathcal{G}(t) := (\mathcal{G}_i(t))_{1 \leq i \leq \ell} \quad (29)$$

as, setting $(e_k)_{1 \leq k \leq n}$ for the canonical basis of $\mathbb{C}^n$

$$\begin{aligned}
 \sum_i \mathcal{G}_i(t) e_i &= \sum_{i < j} G_{i,j}(t) (e_i - e_j) = \sum_{i < j} G_{i,j}(t) e_i - \sum_{j < i} G_{j,i}(t) e_i \\
 &= \sum_{i,j} (G_{i,j}(t) \mathbb{1}_{\{i < j\}} - G_{j,i}(t) \mathbb{1}_{\{j < i\}}) e_i.
 \end{aligned}$$

As a result

$$Q_\lambda(x_1, \dots, x_N | t) =: \frac{1}{(1-t)^{\binom{\ell}{2}}} [U^\lambda] H[(1-t)XU - U^{\varepsilon R}] \mathbb{E}\left(\prod_{i=1}^{\ell} u_i^{\mathcal{G}_i(t)}\right)$$

$$\begin{aligned}
&= \frac{1}{(1-t)^{\binom{\ell}{2}}} \mathbb{E} \left(\left[U^{\lambda - \mathcal{G}(t)} \right] H[(1-t)XU - U^{\varepsilon R}] \right) \\
&=: \frac{1}{(1-t)^{\binom{\ell}{2}}} \mathbb{E}(\tilde{s}_{\lambda - \mathcal{G}(t)}[(1-t)X])
\end{aligned}$$

which gives the result.

• In the R_λ case, one has

$$\begin{aligned}
R_\lambda(x_1, \dots, x_N | t) &:= \sum_{\sigma \in \mathfrak{S}_N} \sigma \cdot \left(X^\lambda H[(1-t)X^{\varepsilon R}] \right) \\
&:= \sum_{\sigma \in \mathfrak{S}_N} \sigma \cdot \left(X^\lambda \prod_{1 \leq k < \ell \leq N} \left(\frac{x_k - tx_\ell}{x_k - x_\ell} \right) \right) \\
&= \frac{(1+t)^{\binom{N}{2}}}{\Delta(X)} \sum_{\sigma \in \mathfrak{S}_N} \varepsilon(\sigma) \sigma \cdot \left(X^\lambda \prod_{1 \leq k < \ell \leq N} \left(\frac{x_k - tx_\ell}{1+t} \right) \right) \\
&= \frac{(1+t)^{\binom{N}{2}}}{\Delta(X)} \sum_{\sigma \in \mathfrak{S}_N} \varepsilon(\sigma) \sigma \cdot \left(X^{\lambda + \delta_N} \prod_{1 \leq k < \ell \leq N} \left(\frac{1 - tx_\ell/x_k}{1+t} \right) \right) \\
&= \frac{(1+t)^{\binom{N}{2}}}{\Delta(X)} \\
&\quad \sum_{\sigma \in \mathfrak{S}_N} \varepsilon(\sigma) \sigma \cdot \left(X^{\lambda + \delta_N} \prod_{1 \leq k < \ell \leq N} \mathbb{E} \left((-x_\ell/x_k)^{B_{k,\ell}(t)} \right) \right)
\end{aligned}$$

where $(B_{k,\ell}(t))_{1 \leq k < \ell \leq N}$ is an i.i.d. sequence of Bernoulli random variables of parameter t .

Define

$$\begin{aligned}
\mathcal{B}_i(t) &:= \sum_{j=1}^{i-1} B_{i,j}(t) - \sum_{j=i+1}^N B_{j,i}(t) \\
\mathcal{B}(t) &:= \sum_{1 \leq i < j \leq N} B_{i,j}(t) \\
\mathcal{B}(t) &:= (\mathcal{B}_i(t))_{1 \leq i \leq N}.
\end{aligned} \tag{30}$$

Using the same manipulations as in the Geometric case, one then obtains

$$\begin{aligned}
R_\lambda(x_1, \dots, x_N | t) &:= \sum_{\sigma \in \mathfrak{S}_N} \sigma \cdot \left(X^\lambda H[(1-t)X^{\varepsilon R}] \right) \\
&= \frac{(1+t)^{\binom{N}{2}}}{\Delta(X)}
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E} \left((-1)^{\sum_{k < \ell} B_{k,\ell}(t)} \sum_{\sigma \in \mathfrak{S}_N} \varepsilon(\sigma) \sigma \cdot \left(X^{\lambda + \delta_N} \prod_{j=1}^n x_j^{\mathcal{B}_j(t)} \right) \right) \\
&= (1+t)^{\binom{N}{2}} \\
& \mathbb{E} \left((-1)^{\mathbf{B}(t)} \times \frac{1}{\Delta(X)} \sum_{\sigma \in \mathfrak{S}_N} \varepsilon(\sigma) \sigma \cdot \left(X^{\lambda - \mathcal{B}(t) + \delta_N} \right) \right) \\
&= (1+t)^{\binom{N}{2}} \mathbb{E} \left((-1)^{\mathbf{B}(t)} \widehat{s}_{\lambda - \mathcal{B}(t)}(X) \right).
\end{aligned}$$

This gives the result. $\square$

3.2. Borodin-Petrov/spin Hall-Littlewood case (Theorem 4)

Proof:

Using (27), one gets

$$\begin{aligned}
\widetilde{F}_\lambda(X|t, (p, \kappa)) &= \sum_{\sigma \in \mathfrak{S}_N} \sigma \cdot \left(H[(1-t)X^{\varepsilon R}] \prod_{j=1}^{\ell(\lambda)} \left(\mathbb{E} \left(x_j^{Z(p, \kappa)} \right) \right)^{\lambda_j} \right) \\
&= \mathbb{E} \left(\sum_{\sigma \in \mathfrak{S}_N} \sigma \cdot \left(H[(1-t)X^{\varepsilon R}] \prod_{j=1}^{\ell(\lambda)} x_j^{\mathcal{Z}_{\lambda_j}^{(j)}(p, \kappa)} \right) \right)
\end{aligned}$$

where

$$\mathcal{Z}_n(p, \kappa) := \sum_{k=1}^n Z_k(p, \kappa)$$

and the sum of random variables defining $\mathcal{Z}_{\lambda_j}^{(j)}(p, \kappa)$ is i.i.d., each such variable being independent.

Defining the vector

$$\mathcal{Z}_\lambda \equiv \mathcal{Z}_\lambda(p, \kappa) := \left(\mathcal{Z}_{\lambda_k}^{(k)}(p, \kappa) \right)_{1 \leq k \leq \ell(\lambda)} \quad (31)$$

one gets

$$\widetilde{F}_\lambda(X|t, (p, \kappa)) = \mathbb{E} (R_{\mathcal{Z}_\lambda(p, \kappa)}(X|t))$$

and using (4), one finally obtains

$$\widetilde{F}_\lambda(X|t, (p, \kappa)) = (1+t)^{\binom{N}{2}} \mathbb{E} \left((-1)^{\mathbf{B}(t)} \widehat{s}_{\mathcal{Z}_\lambda(p, \kappa) - \mathcal{B}(t)}(X) \right)$$

which is the result. $\square$

Remark 6. In [1, § 5], Borodin defines the *stable* spin-Hall-Littlewood polynomials (that pass to the projective limit in n) by

$$F_\lambda(x_1, \dots, x_n | t, s) := \frac{(1-t)^n (1-s)^{-\ell(\lambda)}}{(t; t)_{n-\ell(\lambda)}} \sum_{\sigma \in \mathfrak{S}_n} \sigma \cdot \left(H[(1-t)X^{\varepsilon R}] \prod_{j=1}^{\ell(\lambda)} \frac{1-s}{1-sx_j^{-1}} \left(\frac{x_j-s}{1-sx_j} \right)^{\lambda_j} \right). \quad (32)$$

The proof can of course be adapted with the same random variables up to an additional independent $-G_j(s)$ for an i.i.d. sequence of $\text{Geom}(s)$ random variables using (26), i.e. $\frac{1-s}{1-sx_j^{-1}} = \mathbb{E}\left(x_j^{-G_j(s)}\right)$.

In the same vein, one can deal with the case of (24) using e.g. $n = \ell(\lambda)$ and

$$\begin{aligned} \tilde{G}_\lambda(X|t, s) &= [U^0] H[(1-t)(UX - U^{\varepsilon R})] \\ &\quad \times \prod_{j=1}^{\ell(\lambda)} \mathbb{E}\left(u_j^{G_j(s)-G'_j(s)}\right) \mathbb{E}\left(u_j^{\mathcal{Z}_{\lambda_j}^{(j)}(p, \kappa)}\right) \\ &= \mathbb{E}\left(\left[u_1^{G'_1(s)-G_1(s)-\mathcal{Z}_{\lambda_1}^{(1)}} \dots u_n^{G'_n(s)-G_n(s)-\mathcal{Z}_{\lambda_n}^{(n)}} \right] \right. \\ &\quad \left. \times H[(1-t)(UX - U^{\varepsilon R})] \right) \\ &=: \mathbb{E}\left(\left[U^{\mathcal{G}'(s)-\mathcal{G}(s)-\mathcal{Z}_\lambda} \right] H[(1-t)(UX - U^{\varepsilon R})] \right) \\ &= \mathbb{E}(Q_{\mathcal{G}'(s)-\mathcal{G}(s)-\mathcal{Z}_\lambda}(X|t)) \end{aligned}$$

with obvious notations.

Remark 7. The computations expressing the probabilistic spin-Hall-Littlewood polynomials as expectations of Hall-Littlewood polynomials are similar to the ones of Rozhkovskaya [29, thm. 4.2 (e)]. This last paper uses the formalism of vertex operators for Hall-Littlewood polynomials pioneered by Jing [14]. In the case where the functions f_k defined in [29, (4.9)] are Fourier transforms of discrete random variables, one obtains Theorem 4. The equivalence of computations is given by the fact that the Fourier form the Hall-Littlewood polynomial Q_λ recalled in (19) is in full generality equivalent to a “raising operator” expression (see [21, I-3 p. 42-43] citing [10] and [21, I-5, ex. 29 p. 95]) itself equivalent to an iteration of vertex operators proven by Jing in [14, Prop. 2.17] ; see also the exposition in [21, III-5, ex. 8 p. 236]. Last, see the numerous examples of [29] that enter into the framework of vertex operators (hence Fourier coefficients).

Proof of Corollary 5:

By (21), the eigenvalue of $\mathcal{D}^{(n)}(u|0, t)$ acting on $R_\lambda(X|t)$ is given by $(-u; t)_{n-\ell(\lambda)}$ which is a quantity that only depends on $\ell(\lambda)$. Since $\ell(\lambda) = \ell(\mathcal{Z}_\lambda)$ defined in (31), one can write

$$\begin{aligned}
 \mathcal{D}^{(n)}(u|0, t)\tilde{F}_\lambda(X|t, (p, \kappa)) &= \mathcal{D}^{(n)}(u|0, t)\mathbb{E}(R_{\mathcal{Z}_\lambda}(X|t)) \\
 &= \mathbb{E}\left(\mathcal{D}^{(n)}(u|0, t)R_{\mathcal{Z}_\lambda}(X|t)\right) \\
 &= \mathbb{E}\left((-u; t)_{n-\ell(\mathcal{Z}_\lambda)}R_{\mathcal{Z}_\lambda}(X|t)\right) \\
 &= \mathbb{E}\left((-u; t)_{n-\ell(\lambda)}R_{\mathcal{Z}_\lambda}(X|t)\right) \\
 &= (-u; t)_{n-\ell(\lambda)}\mathbb{E}(R_{\mathcal{Z}_\lambda}(X|t)) \\
 &=: (-u; t)_{n-\ell(\lambda)}\tilde{F}_\lambda(X|t, (p, \kappa))
 \end{aligned}$$

hence the result. $\square$

3.3. Generalisation to symmetric functions defined as $[U^\lambda]H[\mathcal{A}(XU - U^{\varepsilon R})]$

It is of course tempting to generalise (23) to any type of symmetric function defined as $\mathbb{E}(\tilde{s}_{\mathcal{Z}_\lambda - \mathcal{G}}(X))$ for general random variables $\mathcal{Z}$ and $\mathcal{G}$ obtained with a construction similar to theorems 1 and 4. One generalisation of the former concerns symmetric functions that are defined as Fourier coefficients in a similar vein to (16) or (19), i.e.

$$F_\lambda(X|\mathcal{A}) := [U^\lambda]H[\mathcal{A}(XU - U^{\varepsilon R})]. \quad (33)$$

Several such instances are available in the literature ; this is for instance the case of:

- the *Jing symmetric functions* [15, Prop. 1.4 p. 295] where² $\mathcal{A} = \frac{1-t}{1-q}$

$$J_\lambda(X|q, t) := [U^\lambda]H\left[\frac{1-t}{1-q}(XU - U^{\varepsilon R})\right] \quad (34)$$

²Jing also defines a limiting form [15, (1.9) p. 296] with $\mathcal{A} = 1^\alpha$ (defined by setting $t = q^\alpha$ and $q \rightarrow 1$ or $(1, \dots, 1) \in \mathbb{R}^\alpha$ if $\alpha \in \mathbb{N}$) given by $J_\lambda(X|\alpha) := [U^\lambda]H[1^\alpha(XU - U^{\varepsilon R})]$.

- the *Milne symmetric functions* [22, thm. 4.58 p. 224]³ where $\mathcal{A} = (1 - t)(1 - q)$

$$M_\lambda(X|q, t) := \left[U^\lambda \right] H[(1 - t)(1 - q)(XU - U^{\varepsilon R})] \quad (35)$$

- &c.

In each of these cases, one defines, when it is possible, a *discrete plethystic random variable* $\mathcal{H}_d[\mathcal{B}]$ by the equivalent of (26) or (27):

$$H[(z - 1)\mathcal{B}] = \mathbb{E}\left(z^{\mathcal{H}_d[\mathcal{B}]}\right) \quad (36)$$

and theorem 1 can be generalised to this framework under the form

$$F_\lambda(X|\mathcal{A}) = H[\mathcal{A}]^{\binom{\ell(\lambda)}{2}} \times \mathbb{E}(\tilde{s}_{\lambda - \mathcal{G}[1 - \mathcal{A}]}[\mathcal{A}X])$$

where $\mathcal{G}[1 - \mathcal{A}]$ is similar to (29) but with the i.i.d. $G_i(t)$ replaced by i.i.d. copies of $\mathcal{H}_d[1 - \mathcal{A}]$:

$$\begin{aligned} F_\lambda(X|\mathcal{A}) &= \left[U^\lambda \right] H[\mathcal{A}(XU - U^{\varepsilon R})] \\ &= \left[U^\lambda \right] H[\mathcal{A}XU - U^{\varepsilon R}] H[(1 - \mathcal{A})U^{\varepsilon R}] \\ &= H[\mathcal{A}]^{\binom{\ell(\lambda)}{2}} \times \left[U^\lambda \right] H[\mathcal{A}XU - U^{\varepsilon R}] \mathbb{E}\left(U^{\mathcal{G}[1 - \mathcal{A}]}\right). \end{aligned}$$

In particular

- In the Jing case, one has $1 - \mathcal{A} = 1 - \frac{1-t}{1-q} = \frac{t-q}{1-q} = t \frac{1-qt^{-1}}{1-q}$ and the corresponding random variable can easily be proven to be equal in distribution to

$$\mathcal{H}_d\left[t \frac{1 - qt^{-1}}{1 - q}\right] \stackrel{\mathcal{L}}{=} \sum_{k \geq 1} Z(\tau q^k, q^k), \quad \tau := qt^{-1}$$

where Z is defined in (27) and all the random variables in the sum are independent.

³Up to a constant $b_\lambda(t)^{-1}$, the functions defined in [22, (4.59)] are, with our notations, $M_\lambda\left[\frac{X}{1-q} \middle| q, t\right]$.

- In the Milne case, one has $1 - (1-t)(1-q) - 1 = t + q - qt$ and the relevant random variables can be proven to be equal, for instance for $t > q$, to

$$\mathcal{H}_d[t + q - qt] \stackrel{\mathcal{L}}{=} (1 - B(p))G(q) + B(p)G'(q),$$

$$p := \frac{(1-q)^2}{(1-qt)(1-qt^{-1})}$$

where $B(p) \sim \text{Ber}(p)$, $G(t) \sim \text{Geom}(t)$, $G'(q) \sim \text{Geom}(q)$ and these random variables are independent.

Each type of functions admit a rational generalisation by replacing λ by the “rational” random variables $\mathcal{Z}_\lambda$ defined in (31). Of course, other choices of random replacements of λ are possible, and in particular more general rational functions given by the Fourier/Laplace transform of $\mathcal{H}_d[a_1 + \dots + a_p - b_1 - \dots - b_q]$ ($p \geq q$) since $Z(a, b)$ defined in (27) is nothing but $\mathcal{H}_d[b - a]$ or even more general functions given by $z \mapsto \mathcal{H}_d\left[(a_1 + \dots + a_p - b_1 - \dots - b_q) \prod_k \frac{1-t_k}{1-q_k}\right]$.

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